On the selection of preferred consistent sets

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Abstract

The theme of this paper is the multiplicity of the consistent sets emerging in the consistent histories approac to quantum mechanics. We propose one criterion for choosing preferred families among them: that the physically realizable quasiclassical domain ought to be one corresponding to classical histories. We examine the way classical mechanics arises as a particular window, and in the important role played by the canonical group. We finally discuss a possible implication of our selection criterion: that only a class of Hilbert space operators correspond to physical quantities and hence the full non-distributivity of the lattice of quantum propositions is redundant.

The consistent histories approach [1, 2, 3, 4, 5, 6, 7, 8] (or rather approaches taking into account its diverse existing formulations) has been a subject of increasing interest in recent years. It provides a view of time as an intrinsic object in the quantum theory (particularly appealing when considering quantum cosmology), a natural language for discussion of the emergence of classical behaviour and is believed to constitute a converging point for diverse ideas from different interpretations of quantum theory.

Still it seems that an old problem (or at least a point of unease) of the Copenhagen interpretation has been transferred into the histories approach:

the notion of complementarity. The inability according to the Copenhagen interpretation to assign simultaneously properties to a quantum system that correspond to non-commuting operators, is translated in the history language as the question of the many different "windows to reality" or more precisely the multitude of maximal consistent sets of histories corresponding to one decoherence functional. Both are an expression of the non-distributive character of the lattice of quantum propositions; or equivalently from the implicit fundamental assumption that all physical observables (and consequently propositions about them) have a counterpart in Hilbert space operators.

The counter-intuitive (and rather disturbing) nature of the attitude to accept all windows as real has been demonstrated in a series of papers by Kent [9, 10]. He showed, in particular that a) contradictory propositions can be inferred with probability one when one is reasoning within different consistent sets and b) a quasiclassical domain is generically either not stable in time or does not allow inferences according to the predictions of the corresponding classical theory. In absence of a precise selection criterion, the consistent histories theory loses much of its predictive power, since for instance the quasiclassical limit of the non-relativistic classical mechanics cannot be uniquely determined. We thus lose sight of the transition " from Hilbert space to common sense".

In this paper we will argue under the assumption that somehow only one window is physically relevant and realizable , the one corresponding to the classical world as we experience it. [1] There has been a number of ideas and conjectures for an algorithm that would enable us to select this particular window [12, 13], while keeping the established mathematical structures of the history formalism. I think that we should distinguish two possibilities in the formulation of such a criterion:
i) introduce an additional postulate (intrinsic in the mathematical structure of the theory)
ii) specify the physical context of a history theory and identify within this a mechanism that selects a physical window.
We should elaborate on this point in order to clarify the distinction. History theory in its current axiomatic form is of a purely logical - statistical nature. Its primitive concept is the notion of temprorally ordered proposi-

[1] This is not the only option , see for instance [11]

tions, it contains a rule for probability assignement and implication is defined through conditional probability. It is a universal theory since in no point in its axiomatics does the need to specify its physical content arise. In this sense history theory (as any quantum logic scheme) can be viewed as the attempt to capture the general rules, upon which the reasoning about physical systems should be based. And of course, this raises the question (implicit in many debates about the completeness of quantum mechanics) whether it is meaningful to describe the physical world using a context-free abstract language. This debate is particularly pronounced here, since the many windows issue might be taken to imply that not even a concrete representation of the logic can be sufficient to determine the physical content of a theory.

The difference between the two lines of inquiry should now be clear. According to the first, there would be no reason to abandon the universal, logical character of the histories axiomatics, while the second admits the necessity to seek a selection mechanism in the physics of the system.

Proposals of type (i) discussed or implied so far are mainly of a "teleological" [2] type: minimization of some information measure or maximization of some sharp probability measure defined on sets of consistent histories. I do not think that a criterion of this type can be eventually succesful. The reason for this lies in the plethora of consistent sets, many of them being remarkably trivial. For instance, if one wanted to use minimization of the IL entropy as a criterion (which is not something the authors of [13] have proposed but could be a valid conjecture) we would have to contend with histories made out of spectral projections of the initial density matrix. But even if a criterion nicely avoiding all trivial cases were to be found, we would still have to cope with the issue of the non-persistence/ non-predictability of the selected quasiclassical domain. To avoid this one would have to consider sets of histories with temporal support reaching arbitrarily far into the future. This would entail a highly uncausal prediction algorithm, in direct conflict with the (cosmological) motivations of a history theory.

If, on the other hand, we look for a universal physical selection mechanism, we should not have to think much, before conjecturing that the everpresent gravitational field might be somehow relevant. This would be a history language transcription for Penrose's idea of gravitationally induced state vector reduction. Appealing as it might seem, it still has its problems;

[2] In the sense that minimum entropy or minimum action principles are.

if a quantum theory of gravity exists and is to be writen in the history language, what would be its preferred quasiclassical domain? (Or are we to consider quantum gravity as the realm of the many coexistent windows?)

The above remarks have set the conceptual ground, upon which the issues discussed in this paper are to stand. More precisely our aim is as follows: a) propose a natural minimal condition to be satisfied by a preferred window, b) examine a concrete case :the emergence of classical mechanics , c) use the insight obtained by this to strngthen our condition into a sharp selection criterion. The discussion will suggest the existence of preferred variables in quantum mechanics. Finally by inverting the argumentation we will be led to examine the possibility of formulating a history theory with reference only to those objects. Such a version of history theory, would not need to keep the full non-distributivity of the lattice of quantum propositions and would seem to converge to strongly realist interpretational schemes for quantum mechanics.

We start by stating what the minimum requirement for a selection criterion can be and through the study of their consequences we shall phrase we shall arrive at a more precise formulation. Our minimum requirement is that *there is a sense according to which the preferred window retains the temporal structure of the full history theory.* More precisely we want the pre3ferred windowe to correspond to a classical history. [3] By window to reality we define a maximal subset of the algebra of history propositions, such that any exhaustive and exclusive set constructed out of its elements is consistent. Now, this criterion, is an implicit one in Gell-Mann and Hartle's definition of a quasiclassical domain as a maximal consistent subalgebra on which approximately deterministic laws can be defined. Of course this approach is in conflict with ours, since is meant to identify the nature of the window attuned to which an intelligent data gathering and computing system can evolve [4], without doubting the simultaneous validity of the different windows (this has been discussed extensively in [8])

Now we proceed to state more precisely what is involved in the above requirement. For this, one needs first to define what we mean by classical

[3] This requirement seems to be a type (i) one, for it can be formulated solely in terms of primitive elemetns of the theory ; its underlying assumption is that there is no sense in reasoning within windows that do not respect the temporal structure of the full theory. But we shall see that if it is to give a predictable theory a commitment to the physical content (though rather slight) should be made

history theory.

Classical histories
We will prefer to cast the classical histories in the language of classical probability rather than classical mechanics (CM), for the logical structure of the latter is a special case of the former.

The primitive element in probability theory is the notion of the sample space: a measure space X (can be taken as a symplectic manifold in the case of CM). The space of observables is the commutative algebra $O(X)$ of random variables i.e measurable functions from X to $\mathbf{R}$ (usually taken to satisfy some suitable condition according to extra structures on X (e.g. smoothness in the case of CM) A state ω is a positive linear functional on $O(X)$ assigning to each random variable A its mean value $\omega(A)$ (it corresponds to a positive element ρ_ω of $\mathcal{L}^1(X)$.

A single time proposition corresponds to a measurable subset C of X (the system lies in C at this particular time). The set of all measurable subsets of X forms a Boolean lattice $B(X)$, which can be concretely represented by the projective elements of the Banach algebra $\mathcal{L}^\infty(X)$: these are nothing but the characteristic functions $\chi_C(x)$ of the measurable sets. The association of the characteristic functions to propossitions about values of random variables is established through the commutative algebra version of the spectral theorem

$$F(x) = \int \lambda d\lambda \chi_{C_\lambda}(x) \tag{1}$$

where $F(x)$ a general element of $O(X)$ and C_λ is the subset of X where F takes value λ.

It is easy to see how one can translate the constructions of quantum mechanical histories in this case. An n-time history is essentially a string of measurable subsets of x: $(C_{t_1}, C_{t_2}, \ldots, C_{t_n})$. This is represented by the projector $\chi_{C_1} \otimes \ldots \otimes \chi_{C_n}$, which is an element of $\otimes_n \mathcal{L}^\infty(X) = \mathcal{L}(\times_n X)$. The construction of the space of history propositions then trivially proceeds along the lines of Isham [6]. [4]

In a classical theory there is no obstacle in consistently assigning probability measures to all histories. Nevertheless, we will try to write the probability

[4] The construction for continuous-time histories should follow similar reasoning: taking the quantities $x_t \in \times_t X_t$ as corresponding to fine grained histories, and constructing coarse-grainings through the use of a Wiener measure on X.

assignment through a decoherence functional, to establish contact with the quantum case. Of course, the interference measured by this will be trivial, i.e. it will be due only to the non-disjointness of a pair of histories.

Time evolution in a classical probabilistic theory is given by a family (actually a semigroup) of bistochastic maps [5] $T_t : \mathcal{L}^\infty(X) \to \mathcal{L}^\infty(X)$. These induce a family of maps $T_t^\dagger$ on the space of states by $(T_t^\dagger \omega)(A) = \omega(T(A))$, in terms of which a (real-valued) decoherence functional for a pair of histories $\alpha = (C_1, \ldots, C_n)$ and $\alpha' = (C_1', \ldots, C_n')$ with same temporal support, reads

$$d(\alpha, \alpha') = Tr\left(\chi_{C_n}(T^\dagger_{t_n - t_{n-1}}(\ldots T^\dagger_{t_2 - t_1}(\chi_{C_1} T^\dagger_{t_1}(\rho_0))\chi_{C_1'})\ldots))\chi_{C_n'}\right) \qquad (2)$$

We should distinguish two cases:
i) Deterministic dynamics: T is an automorphism of the algebra of observables and generically can be defined through a permutation τ on X (a canonical transformation for the case of CM): $T(A)(x) = \tau^* A(x) = A(\tau x)$. In this case it is trivial to verify that a history can be represented by a projector on $\mathcal{L}^\infty(X)$, namely

$$\chi_\alpha = \chi_{\tau_1 C_1 \cap \tau_2 C_2 \cap \ldots \tau_n C_n} \qquad (3)$$

ii) Random dynamics: T is a convex combination of automorphisms: $T = \sum_i \lambda_i \tau_i^*$. Using the convexity property of the decoherence functionals [13], it is easy to establish that expression (3) defines a well behaved decoherence functional. [6]

Finally we choose to define implication through conditional probability. If α and β are disjoint propositions then we say that α *implies* β if $p(\alpha \cup \beta) = p(\alpha)$.

We could then proceed and incorporate the classical history theory as a special case of the history axiomatics as written down by Isham [6], except for the fact that not every Boolean lattice is suitable as a space of history

[5] A linear, positive map $T : \mathcal{L}^\infty(X) \to \mathcal{L}^\infty(X)$ is called bistochastic if $T(1) = 1$ and its restriction on $\mathcal{L}^1(X)$ is trace preserving.

[6] Incidental to our aims, is the fact that this construction of a decoherence functional containing random dynamics can be repeated (modulo a few technicalities) in the quantum mechanical case by making use of the ILS theorem [15]. In particular this implies that theories of stochastic state vector reduction (like in [16]) can be nicely incorporated in the history formalism, and hence suffer from the multiple windows problem. We can therefore conclude that no modification in the dynamics, is sufficient by itself to provide a selection criterion.

propositions. Fundamental for any formulation of probability theory is the stability of the sample space through time. Hence the lattice of history propositions should always be identified with the characteristic functions of some measurable space of the form $\times_t X_t$, where the X_t are related by structure-preserving bijections.

The classical mechanics window

We are now going to examine the sense in which the world of classical mechanics arises as a particular window in a quantum mechanical history theory . The aim is to identify the important structures that eventually give the corresponding quasiclassical domain and use this information to sharpen our previous requirements to a selection criterion. We will restrict ourselves to the case of non-relativistic particle mechanics, since a) for quantum field theory neither do we have a history version, nor do we have a clear understanding of its classical limit, b) we shall rely on the insight obtained by Omnés semiclassical theorems [3].

Let us anticipate the discussion to follow and give a useful characterization for a class of classical histories. Whenever the sample space carries a metric structure one can define a map $\nu : \mathcal{B}(X) \to \mathbf{R}^+$, such that $\nu(C)$ gives a measure of the size and regularity of the cell C (i.e the ratio of the area of its boundary ∂C to its volume), in such a way that $\nu(C)$ is close to zero when C is sufficiently large and regular. There is a precise sense in which ν can be constructed; the reader is referred to Omnés [3] for details. We then call a classical history theory $(\times_t X_t, d, \mu)$ ϵ-deterministic, if for any cell C such that $\nu(C) = O(\epsilon)$ there exists another cell C' with $\nu(C') = O(\epsilon)$ such that the conditional probability $p(C, t_1; C', t_2 | C, t_1)$ is of order $O(\epsilon)$. This essentially means that the randomness of the dynamics is significant only on scales of the order of the characteristic length of the metric. Two ϵ- deterministic theories are equivalent if they have isomorphic sample space X and the conditional probabilities of the two theories differ in the order of ϵ.

Omnés' construction starts from the well-known fact that the classical phase space can be naturally embedded in the projective space of its corresponding Hilbert space by the use of the canonical group. Starting from the representation of the canonical group on the Hilbert space of the theory, one can construct the mapping j_r from the phase space Γ to the coherent state projectors $z \to P_z = U(z)|r\rangle\langle r|U(z)^{-1}$, where $|r\rangle$ an arbitrary vector of H. Hence j_r embeds X into $\mathcal{P}H$. The pull-back of the Fubini-Study metric on

$\mathcal{P}H$ defines a metric on X , with respect to which a function ν of the type discussed above can be constructed. Note, that for a generic group there is an optimization algorithm [17] for the choice of $|r\rangle$, so that the characteristic scale of the metric on X is essentially $\hbar$. In the case of quantum mechanics on $\mathcal{L}^2(\mathbf{R}^n)$ this algorithm leads to Gaussian coherent states.

One can then construct approximate projectors corresponding to "classical" type of propositions about phase space cells:

$$P_C = \int_C d\mu(z) P_z \tag{4}$$

where μ is a measure on X. If $\nu(C) = O(\epsilon)$, then any projector in an ϵ-neighbourhood of P_z can be thought as representing the cell C.

To establish consistency one needs then an important fact: that coherent states on $\mathcal{L}^2(\mathbf{R}^n)$ are approximately stable under time evolution generated by Hamiltonians of the form $H = \frac{p^2}{2m} + V(x)$ for a large class of physically interesting potentials $V(x)$. This can be used to establish consistency for all exclusive and exhaustive sets of histories constructed out of quasiprojectors corresponding to cells C with small value of ν. [7] We also get approximate determinism, corresponding to evolution with a classical Hamiltonian $H(z) = Tr(P_z H)$. In this sense the window correponding to classical mechanics corresponds to a classical ϵ-deterministic history theory (by no means a unique one).

[7] What Omnés has actually established is approximate consistency (of order $\epsilon = \nu(C)$) of these sets, and assumed that there exist history propositions close to them, out of which an exact consistent set can be formed. Although this is a plausible assumption, it still remains to be proved. Let us see what is involved in this proof. Taking two n-time histories the decoherence functional can be restricted to a continuous map d : $(\otimes_n B(H)) \otimes (\otimes_n B(H)) \rightarrow \mathbf{C}$. Consider a set of N history propositions constructed by approximate projectors P_i, $i = 1 \dots N$, and assume $d(P_i, P_j) < \epsilon$ (an unsharp approximate consistency criterion). Now if one defines the sets $O_\epsilon = d^{-1}(\{z \in C; |z| < \epsilon\})$ and $C_0 = d^{-1}(\{0\})$, it is sufficient to show that the connected component of C_0 contained in the same component of O_ϵ with $P_i \otimes P_j$ has non-empty interior. For then we can use the fact that the set of projectors is dense in $B(H)$ to associate exact projectors P_j^{exact} in as ϵ-neighbourhood of P_j such that we have exact consistency. This seems to a generic case for sets of $n - time$ histories in infinite dimensional Hilbert spaces. Of course, when one assumes continuous time histories the issue becomes more complicated. A study of the general case would be invaluable towards understanding the "predictability sieve" on the possible fine-grainings of a consistent set.

From the above discussion it is clear that the classical mechanics window satisfies our previously stated requirement. We are now in a position to give a more precise characterization : *A preferred window such that each consistent subset of it can be embedded into the lattice of propositions of a classical history theory.* It might be the case that to get uniqueness, we must impose some condition of maximality on the corresponding classical history ,so that trivial windows will get excluded. An example is the window corresponding to propositions about values of energy, any subset of which is trivially consistent (it seems that one cannot get any contradictory inferences by considering such a set in conjunction with the CM window). Anyway, if it turns out to be necessary to include a maximality condition, this can easily take the form of specifying an extra structure (i.e. the requirement that it is a topological space or a manifold for the case of CM) for the sample space X of the emergent (or underlying?) classical history theory.

In our particular case, the window corresponding to classical mechanics can be shown to be essentially unique. For a different selected window (corresponding to a classical history with sample space X), would imply the existence of an embedding of X on the projective Hilbert space (or more generally the existence of a projection-valued measure (PVM) on X. Without loss of generality we can take X to be a subspace of $\mathbf{R}^n$, and consider the marginal PVM corresponding to an one-dimensional submanifold of X. This defines a self-adjoint operator. Now a fundamental property of the canonical group is that its spectral projections generate the whole of $B(H)$ and hence this operator can be represented as $f(\hat{z})$, where $\hat{z}$ represents the generators of the canonical group and f some measurable function . This means that the classical history propositions corresponding to X can be embedded into the lattice of history propositions of the CM window. [8] It is in this sense that the CM window (assuming of course that it exists) is maximal. (Note that in the above argumentation X is assumed to be a manifold). A remark is in order at this point concerning the role of the hydrodynamic variables discussed extensively by Gell-Mann and Hartle. Whenever the CM-window exists, the construction of theses variables proceeds along the lines of classical

[8] There are factor ordering ambiguities when choosing f. To make the above arguments into a rigorous proof one needs to show that observables corresponding to fucntions with different factor ordering define equivalent ϵ-deterministic thepories. This is quite plausible for ϵ in the order of $\hbar$ in some power in view of already existent semiclassical theorems , but we have been unable to find a general proof

hydrodynamics, since they can be viewed as further coarse graining on the emergent classical history theory. More interesting would be the case where the CM-window does not exist (i.e. the propositions constructed from the canonical group form no non-trivial consistent sets). In that case it is conceivable that two complementary hydrodynamic windows might exist. This would of course invalidate our argument that the proposed selection criterion chooses a unique window. (It is hard to imagine such a set of variables that does not reduce to ordinary hydrodynamics or at least to an extension of irreversible thermodynamics). The physical system where this might be possible is quantum field theory, where there are indirect arguments [18] (though definitely not conclusive) that the classical field theory window might not be emerging for a large class of states of the system.

Preferred window and non-distributivity

From the arguments stated earlier, it should be clear that *if we commit ourselves to the logic of one preferred window*, we cannot escape the conclusion that they are generated by a particular class of Hilbert space operators (the ones determined by the canonical group). The question then arises: what is the nature of the other windows (the ones not stable in time)? They cannot be considered as anything but redundant, for they are irrelevant to any physical predictions of the theory. But then, why do they appear at all in the formalism? The only possible answer is that there is a redundancy in the primitive elements of the history theory. Not all history propositions can be considered as physical. This is a point strongly reminiscent of the one advocated in the context of single-time quantum mechanics by Margenau and Park [19]: the incomeasurability of observables corresponding to non-commuting operators can be true in quantum mechanics *only if accept as an axiom that any operator on the Hilbert space corresponds to some physical observable.* The many windows problem is just the history counterpart of the incomeasurability problem of single-time quantum mechanics; and it is due to the richest structure of the former that the counterintuitive effects of such a postulate are most impressive. [9]

A consequence of not accepting all operators as physical, is that there

[9] Actually one of the strongest arguments posed by Margenau and Park, was the time-of-flight type of experiments, with an analysis that can be thought as an ancestor of the history formalism

is no need to keep the non-distibutive character of the lattice of history propositions (see [19] for extensive discussion); indeed reasoning inversely, the non-distibutivity is the main assumption that allows multiple windows. One should then start thinking of possible substitutes. If one opts for a distributive lattice structure for history propositions, the first choice would be to consider Boolean lattices (this is not the only possible choice, see for instance [20]). Of course, hidden variables theories fall in this category (even though they would correspond to a classical history theory). Another approach would be Sorkin's quantum measure theory, where the lattice of history propositions is assumed Boolean and the role of the decoherence functional is played bya non-additive measure on this lattice [21].

It turns out that a history theory with a preferred window looks similar to a Sorkin-type construction. Recall the importance of the canonical group towards identifying the preferred window. In standard quantum mechanics, its role is also important. Since all infinite dimensional Hilbert spaces are isomorphic, the only way one can separate the physical content of (say) $\mathcal{L}^2(\mathbf{R})$ from $\mathcal{L}^2(\mathbf{R}^3)$ is by considering the representations of different canonical groups (its history version is also very important towards constructing explicitly the space of history propositions). One is then tempted to the question? Can one construct a history theory using only "classical" primitive elements (canonical group or the phase space that can be constructed from it, possibly plus some additional (complex) structure)? One can for instance assume as primitive elements the paths $z(t), z^*(t)$ on some "phase space" X and their coarse grainings as given by integration with respect to some Wiener measure and a coherent-state path-integral version of the decoherence functional between pairs of coarse grained histories $\alpha = \int_{C_\alpha} d\mu(z(.), z^*(.))$ and $\alpha' = \int_{C'_\alpha} d\mu(z(.), z^*(.))$

$$d(\alpha, \alpha') = \int dz_f dz_f^* dz_i dz_i^* dz_i^* dz_i' dz_i'^* e^{-z_f^* z_f - z_i^* z_i - z_i'^* z_i'}$$
$$\left(\int_{C_\alpha} d\mu(z(.), z^*(.)) \int_{C_{\alpha'}} d\mu(z'(.), z'^*(.)) e^{iS[z(.),z^*(.)] - iS[z'(.),z'^*(.)]} \right) \rho(z_i^*, z_i') \quad (5)$$

with path integration over paths such that $z(0) = z_i$, $z^*(t) = z_f^*$, $z'(t) = z_f$, $z'^*(0) = z_i'^*$. Modulo some difference in the probability assignment this construction is a close relative of Sorkin's theory.

Unfortunately, this construction is at least incomplete and for a more fundamental reason than the proverbial non-definability of the path integral

with Wiener measures: there is no way we can reproduce the quantum mechanical combination of subsystems via the tensor product solely from the knowledge of X; we have to introduce a linear structure and hence the Hilbert space would eventually enter again our schemes.

Before concluding let us summarize the thesis of this paper. I one wishes for a history theory allowing a) maximum predictability and b) a realist commitment, the option of seeking a selection algorithm among different windows is a natural one. Our proposed selection criterion (of temporal stability of the preferred window) seems to imply that only a class of history propositions is physically relevant, hence that the non-distributivity of the lattice of history propositions is redundant. If one then attempts to write an "economic" version of history theory, where only physical quantities are taken into account, then one faces one fundamental problem: how would the Hilbert space structure emerge in such a theory. The problem can be neatly summarized as the inverse of the many windows problem: How does one recover the Hilbert space from (distributive) "common sense"?

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